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1992 J. Phys. A: Math. Gen. 25 877

(http://iopscience.iop.org/0305-4470/25/4/024)

J. Phys. A: Math. Gen. 25 (1992) 877–883. Printed in the UK

Electromagnetic cascades: an alternative solution*

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Received 31 May 1991, in final form 10 October 1991

Abstract. We solve the one-dimensional diffusion equations for the electromagnetic cascade shower in a spinorial representation.

1. Introduction

The electromagnetic cascade shower is a multiplicative process involving the interactions of electrons and photons when passing through matter. The description of the electromagnetic cascade can be formulated mathematically as a system of simultaneous differential equations for the number of electrons and photons at a certain depth of matter t, with energies between E and $E + \mathrm{d}E$. A well-known solution for these differential equations was obtained by Rossi and Greisen [1], based on functional transforms.

In this paper we present an alternative way to obtain the solution for these differential equations based on a spinorial representation, using a Feynman-like procedure of ordered exponential operators [2, 3] to take into account the multiple interactions with matter. We solve the one-dimensional differential equations under two approximations, traditionally named A and B. 'Approximation A' stands for electrons and photons with large energies compared with the critical energy of the traversed material, so that the ionization loss of electrons and the effect of Compton scattering can be neglected. This situation is treated in section 2. If the effect of ionization loss is included, but the Compton scattering is still neglected, the treatment is called 'approximation B', as considered in section 3. In both cases the solution is obtained using complete-screening cross sections for radiation and pair creation processes [4].

2. Solution in 'approximation A'

Let $\Pi(E,t)\mathrm{d}E$ and $\gamma(E,t)\mathrm{d}E$ be the average number of electrons and photons with energies between E and $E + \mathrm{d}E$ in a shower at depth t (in radiation lengths).

* Partial financial support of FAPESP and CNPq, Brazil.

0305-4470/92/040877+07\$04.50

In approximation A, the diffusion equations are

$$\frac{\partial}{\partial t}\Pi(E,t) = -\hat{A}\Pi(E,t) + \hat{B}\gamma(E,t)$$
$$\frac{\partial}{\partial t}\gamma(E,t) = \hat{C}\Pi(E,t) - \hat{D}\gamma(E,t) \tag{1}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are operators defined by

$$\begin{aligned}
-\hat{A}\Pi(E,t) &= -\int_0^1 \left\{\Pi(E,t) - \frac{1}{1-v}\Pi\left(\frac{E}{(1-v)},t\right)\right\}\varphi_0(v)\,\mathrm{d}v \\
\hat{B}\gamma(E,t) &= 2\int_0^1 \gamma(E/u,t)\psi_0(u)\,\mathrm{d}u/u \\
\hat{C}\Pi(E,t) &= \int_0^1 \Pi(E/v,t)\varphi_0(v)\,\mathrm{d}v/v \\
-\hat{D}\gamma(E,t) &= -\int_0^1 \gamma(E,t)\psi_0(u)\,\mathrm{d}u \equiv -\sigma_0\gamma(E,t)
\end{aligned} \tag{2}$$

and $\varphi_0(v)\mathrm{d}v = \varphi_0(E,E')\mathrm{d}E'/E$ is the probability, per radiation length, of the emission of a photon in the energy interval $(E', E'+\mathrm{d}E')$ by an electron of energy E, and $\psi_0(u)\mathrm{d}u = \psi_0(E,E')\mathrm{d}E'/E$ is the probability, per radiation length, of pair creation by a photon with energy E, giving rise to an electron in the interval $(E', E'+\mathrm{d}E')$ [1, 4].

In order to solve (1) we define a two-component spinor

$$\phi(E,t) = \begin{pmatrix} \Pi(E,t) \\ \gamma(E,t) \end{pmatrix} \tag{3}$$

and a 2×2 operator matrix

$$\hat{\Sigma} = \begin{pmatrix} -\hat{A} & \hat{B} \\ \hat{C} & -\hat{D} \end{pmatrix}. \tag{4}$$

Equations (1) can be transformed in a spinor differential equation

$$\frac{\partial}{\partial t}\phi(E,t) = \hat{\Sigma}\phi(E,t). \tag{5}$$

When the electromagnetic cascade shower is initiated at depth $t = 0$ by a particle with defined energy E_0, the boundary condition of (5) can be written as

$$\phi(E,0) = \delta(E-E_0)\phi_0 \tag{6}$$

where the two-component spinor ϕ_0 depends on the particular considered cascade. If the cascade is initiated by an electron, then $\phi_0 = \binom{1}{0}$, otherwise $\phi_0 = \binom{0}{1}$, when it is initiated by a photon.

Using the Mellin integral representation for $\delta(E - E_0)$, we have as a boundary condition

$$\phi(E,0) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}s \left(\frac{E_0}{E}\right)^s \frac{1}{E}\phi_0. \tag{7}$$

The formal solution of (5) is given by

$$\Phi(E, E_0, t) = \mathrm{Exp}\left[\int_0^t \hat{\Sigma}\,\mathrm{d}t'\right]\Phi(E,0) \tag{8}$$

where $\mathrm{Exp}[\int_0^t \hat{\Sigma}\,\mathrm{d}t']$ is an expansional, defined by a sum of multiple depth ordered integrals [2, 3], so that

$$\Phi(E, E_0, t) = \left[I + \int_0^t \hat{\Sigma}\,\mathrm{d}t' + \int_0^t \hat{\Sigma}\,\mathrm{d}t' \int_0^{t'} \hat{\Sigma}\,\mathrm{d}t'' + \cdots\right]\Phi(E,0). \tag{9}$$

Once the $\hat{\Sigma}$-matrix acts only on the energy dependence of $\Phi(E,0)$, which is manifestly t-independent, it is straighforward to rewrite (9) as

$$\begin{aligned}\Phi(E, E_0, t) = \Big[&I\Phi(E,0) + t\hat{\Sigma}\Phi(E,0) + \frac{t^2}{2!}\hat{\Sigma}\hat{\Sigma}\Phi(E,0) \\ &+ \frac{t^3}{3!}\hat{\Sigma}\hat{\Sigma}\hat{\Sigma}\Phi(E,0) + \cdots\Big]\end{aligned} \tag{10}$$

which can be expressed simply by

$$\Phi(E, E_0, t) = \exp[t\hat{\Sigma}]\Phi(E,0). \tag{11}$$

The 2×2 matrix $t\hat{\Sigma}$ can be written in terms of the Pauli matrices

$$t\hat{\Sigma} = \hat{x}_0 I + \boldsymbol{x} \cdot \boldsymbol{\sigma} \tag{12}$$

where

$$\begin{aligned} \hat{x}_0 &= -(t/2)(\hat{A} + \hat{D}) & \hat{x}_1 &= (t/2)(\hat{B} + \hat{C}) \\ \hat{x}_2 &= \mathrm{i}(t/2)(\hat{B} - \hat{C}) & \hat{x}_3 &= -(t/2)(\hat{A} - \hat{D}). \end{aligned} \tag{13}$$

Since $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are commutative operators, then $(\boldsymbol{x} \cdot \boldsymbol{\sigma})^2 = \boldsymbol{x} \cdot \boldsymbol{x}$ and we can introduce the following operator

$$\hat{w} \equiv (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2} = (t/2)\sqrt{(\hat{A} - \hat{D})^2 + 4\hat{B}\hat{C}}. \tag{14}$$

Hence, with the previous notation, the exponential operator $\exp[t\hat{\Sigma}]$ in (11) can be rewritten as

$$\exp[t\hat{\Sigma}] = \mathrm{e}^{\hat{x}_0}\left\{I\cosh\hat{w} + (\boldsymbol{x} \cdot \boldsymbol{\sigma})\frac{\sinh\hat{w}}{\hat{w}}\right\}. \tag{15}$$

We note that by expanding the right-hand side of (15) we get only even powers of $\hat{w}$. This justifies the use of the notation used in (14).

Applying (15) to the boundary condition, (7), we obtain the solution of (5)

$$\phi(E, E_0, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{E_0}{E}\right)^s \frac{1}{E} M_0(s,t)\phi_0 \tag{16}$$

where $M_0(s,t)$ is a 2×2 matrix defined by

$$M_0(s,t) = e^{x_0(s)}\left\{I \cosh w(s) + (\boldsymbol{x}(s)\cdot\boldsymbol{\sigma})\frac{\sinh w(s)}{w(s)}\right\}. \tag{17}$$

Here, $x_0(s), \boldsymbol{x}(s)$ and $w(s)$ are, respectively, the eigenvalues of the operators $\hat{x}_0, \boldsymbol{x}$ and $\hat{w}$, in this energy basis. Formally, this means replacing the operators $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ in (13) and (14) by their eigenvalues [1, 4]

$$\begin{aligned}
A(s) &= \int_0^1 [1-(1-v)^s]\varphi_0(v)\,dv \\
B(s) &= 2\int_0^1 u^s \psi_0(u)\,du \\
C(s) &= \int_0^1 v^s \varphi_0(v)\,dv \\
D(s) &= \sigma_0.
\end{aligned} \tag{18}$$

In a more concise way, (17) can be written as

$$M_0(s,t) = e^{t\Sigma(s)}. \tag{19}$$

For a given choice of the two-component spinor ϕ_0, which defines the primary particle initiating the cascade shower, approximate methods must be applied to the integral in equation(16), in order to evaluate the solution $\phi(E, E_0, t)$.

In order to see that (16) and (17) are exactly the solutions by Rossi and Greisen [1] we express $M_0(s,t)$ in the following way

$$M_0(s,t) = \begin{pmatrix} \Pi_e(s,t) & \Pi_\gamma(s,t) \\ \gamma_e(s,t) & \gamma_\gamma(s,t) \end{pmatrix} \tag{20}$$

so that the first column of the matrix corresponds to the cascade initiated by an electron and the second column to one initiated by a photon.

Rewriting also (17) in the 2×2 matrix form and identifying it with (20), term by term, we obtain:

$$\begin{aligned}
\Pi_e(s,t) &= \frac{x_3(s)+w(s)}{2w(s)}e^{x_0(s)+w(s)} - \frac{x_3(s)-w(s)}{2w(s)}e^{x_0(s)-w(s)} \\
\gamma_e(s,t) &= \frac{tC(s)}{2w(s)}(e^{x_0(s)+w(s)} - e^{x_0(s)-w(s)}) \\
\Pi_\gamma(s,t) &= \frac{tB(s)}{2w(s)}(e^{x_0(s)+w(s)} - e^{x_0(s)-w(s)}) \\
\gamma_\gamma(s,t) &= -\frac{x_3(s)-w(s)}{2w(s)}e^{x_0(s)+w(s)} + \frac{x_3(s)+w(s)}{2w(s)}e^{x_0(s)-w(s)}.
\end{aligned} \tag{21}$$

Following the notation of Nishimura [4], and using the eigenvalues corresponding to (13) and (14), we define

$$\lambda_{1,2}(s) \equiv \frac{x_0(s) \pm w(s)}{t} = -\frac{A(s) + \sigma_0}{2} \pm \frac{1}{2}\sqrt{(A(s) - \sigma_0)^2 + 4B(s)C(s)} \quad (22)$$

and

$$H_{1,2}(s) \equiv \frac{x_3(s) \pm w(s)}{2w(s)} = \pm\frac{\sigma_0 + \lambda_{1,2}(s)}{\lambda_1(s) - \lambda_2(s)} \quad (23)$$

where the subscript 1(2) in λ and H corresponds to the upper (lower) signs. We note also that $\lambda_1(s) - \lambda_2(s) = 2w(s)/t$, so that (21) becomes

$$\begin{aligned} \Pi_e(s,t) &= H_1(s)e^{\lambda_1(s)t} + H_2(s)e^{\lambda_2(s)t} \\ \gamma_e(s,t) &= \frac{C(s)}{\lambda_1(s) - \lambda_2(s)}\{e^{\lambda_1(s)t} - e^{\lambda_2(s)t}\} \\ \Pi_\gamma(s,t) &= \frac{B(s)}{\lambda_1(s) - \lambda_2(s)}\{e^{\lambda_1(s)t} - e^{\lambda_2(s)t}\} \\ \gamma_\gamma(s,t) &= H_2(s)e^{\lambda_1(s)t} + H_1(s)e^{\lambda_2(s)t} \end{aligned} \quad (24)$$

which reproduces the solutions obtained by Rossi and Greisen [1] in Nishimura's notation.

3. Solution in 'approximation B'

The diffusion equations are now given by

$$\begin{aligned} \frac{\partial}{\partial t}\Pi(E,t) &= -\hat{A}\Pi(E,t) + \hat{B}\gamma(E,t) + \varepsilon\frac{\partial}{\partial E}\Pi(E,t) \\ \frac{\partial}{\partial t}\gamma(E,t) &= \hat{C}\Pi(E,t) - \hat{D}\gamma(E,t) \end{aligned} \quad (25)$$

where ε is the critical energy for the effect of ionization loss by electrons. In the spinorial form these equations can be written in the following way:

$$\frac{\partial}{\partial t}\phi(E,t) = \hat{\Sigma}\phi(E,t) + \hat{\Sigma}_1\phi(E,t) \quad (26)$$

where $\hat{\Sigma}$ is given by (4), $\hat{\Sigma}_1$ is the 2×2 operator matrix

$$\hat{\Sigma}_1 = \frac{\varepsilon}{2}\frac{\partial}{\partial E}(I + \sigma_3) \quad (27)$$

and the boundary condition is expressed by (7).

Since $\hat{\Sigma}$ and $\hat{\Sigma}_1$ are non-commutative operators, the formal solution of (26) is given in terms of expansionals by

$$\phi(E, E_0, t) = \mathrm{Exp}\left[\int_0^t (\hat{\Sigma} + \hat{\Sigma}_1)\,\mathrm{d}z\right]\phi(E,0). \quad (28)$$

According to the Feynman-like formalism of ordered exponential operators, the expansional in (28) can be decomposed as follows [2, 3]:

$$\operatorname{Exp}\left[\int_0^t (\hat{\Sigma}+\hat{\Sigma}_1)\,\mathrm{d}z\right] = \operatorname{Exp}\left[\int_0^t \hat{O}(t,z)\hat{\Sigma}_1\hat{O}^{-1}(t,z)\,\mathrm{d}z\right]\hat{O}(t,0) \tag{29}$$

with

$$\hat{O}(t,z) = \operatorname{Exp}\left[\int_z^t \hat{\Sigma}\,\mathrm{d}z'\right] = \exp[(t-z)\hat{\Sigma}]. \tag{30}$$

The expansional in (30) reduces itself to an ordinary exponential, as shown in approximation A.

From now on, our procedure will be to calculate the leading terms of the expansional (29) first, and then, from these terms, find a generalization to the complete solution (28):

$$\left[1+\int_0^t \hat{O}(t,z)\hat{\Sigma}_1\hat{O}^{-1}(t,z)\,\mathrm{d}z\right]\hat{O}(t,0) = \hat{O}(t,0)+\int_0^t \hat{O}(t,z)\hat{\Sigma}_1\hat{O}(z,0)\,\mathrm{d}z. \tag{31}$$

The first term on the right-hand side of (31) is the zeroth order of (29) and leads to the solution already obtained under 'approximation A'. The contribution of the second term, when applied to (28), gives

$$\int_0^t \mathrm{d}z\,\hat{O}(t,z)\hat{\Sigma}_1\hat{O}(z,0)\phi(E,0) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\left(\frac{E_0}{E}\right)^s\frac{1}{E}\left(-\frac{\varepsilon}{E}\right)(s+1)$$
$$\times\int_0^t \mathrm{d}z\,\mathrm{e}^{(t-z)\Sigma(s+1)}\tfrac{1}{2}(I+\sigma_3)\mathrm{e}^{z\Sigma(s)}\phi_0. \tag{32}$$

Using the notation of (19), we have

$$\exp[(t-z)\Sigma(s+1)] = M_0(s+1,t-z)$$

and so the solution $\phi(E,E_0,t)$, up to first order in (ε/E), is

$$\phi(E,E_0,t) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\left(\frac{E_0}{E}\right)^s\frac{1}{E}\left\{M_0(s,t)+\left(\frac{-\varepsilon}{E}\right)(s+1)\right.$$
$$\left.\times\int_0^t \mathrm{d}z\,M_0(s+1,t-z)\tfrac{1}{2}(I+\sigma_3)M_0(s,z)\right\}\phi_0. \tag{33}$$

It is straightforward to calculate the contribution of the higher-order terms of the expansional, (29), and write the spinor solution as

$$\phi(E,E_0,t) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\left(\frac{E_0}{E}\right)^s\frac{1}{E}\left\{M_0(s,t)+\sum_{n=1}^{\infty}\left(-\frac{\varepsilon}{E}\right)^n M_n(s,t)\right\}\phi_0 \tag{34}$$

where the 2×2 matrix $M_n(s,t)$ is derived from the following recurrence relationship:

$$M_n(s,t) = (s+n)\int_0^t M_0(s+n,t-z)\tfrac{1}{2}(I+\sigma_3)M_{n-1}(s,z)\,\mathrm{d}z. \tag{35}$$

It is easy to see that (34) and (35) reproduce the particular solutions obtained by Nishimura [4]. In his work, the behaviour of (34) is also discussed, giving the result that the series is uniformly convergent if $2\varepsilon t/E < 1$. Anyway, in the high energy region the series is always uniformly convergent and the solution (34) with (35) can be approximated by (33).

4. Conclusion

In conclusion, we have calculated the solutions of the one-dimensional diffusion equations for the electromagnetic cascade showers in approximations A and B, within a spinorial formalism. We have shown that, using a Feynman-like formalism of ordered exponential operators, the solution in approximation B can easily be calculated, if we know the solution under approximation A. The great advantage of this method lies in its simplicity. Although we have obtained the same results already obtained by Rossi and Greisen [1], and Nishimura [4], the method allows the results to be obtained in a straightforward way, without any previous supposition on the behaviour of the solutions.

This procedure can also be applied to obtain the solutions in other approximations [4], particularly to analyse the solutions of the three-dimensional diffusion equations.

Acknowledgments

The authors would like to thank FAPESP (CGSC and LMMF) and CNPq (JBF), Brazil, for financial support.

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